Tethering Vertical Merger Analysis – Technical Notes
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These notes provide analytical support for the points in the paper "Tethering Vertical Merger Analysis."

Formal Derivation of Foreclosure Arithmetic

This section derives standard foreclosure math and provides a rigorous definition of the departure rate, showing how it naturally breaks into two factors: the diversion ratio, which is a property of the final demand system, and the departure rate, which depends on the intensity of upstream competition.

An upstream firm facing a competitive fringe (unmodeled) sells to two downstream firms, labled 1 and 2. The upstream firm merges with downstream firm 1. The integrated firm's profit is

$$\pi = (p_1 - c_1 - v_1)D_1 + (w_2 - c_2)D_2^d$$

where

- p_1 is the integrated firm's downstream price,
- c_1 is the integrated firm's marginal cost of producing the input internally,
- c_2 is the integrated firm's marginal cost of producing an input and selling it to the rival,
- v_1 is the integrated firm's marginal cost of output due to factors other than the input,
- w_2 is the wholesale price,
- D_1 is the integrated firm's sales of the downstream product, and
- D_2^d is the sale of the input to the rival (the residual derived demand for the integrated firm's input).

Upstream and downstream quantities vary in fixed proportions and are measured in the same units.

Holding prices fixed, the change in the integrated firm's profit in response to some foreclosure strategy is

$$\Delta\pi = (p_1 - c_1 - v_1)\Delta D_1 + (w_2 - c_2)\Delta D_2^d.$$

After some rearrangement,

$$\Delta\pi > 0 \qquad \text{if and only if} \qquad -\left(\frac{\Delta D_1}{\Delta D_2^d}\right) > \frac{M_{u2}}{M_I}$$

where $M_{u2} = w - c_2$ is the unit margin on sales of the input to the rival and $M_I = p_1 - c_1 - v_1$ is the integrated unit margin. We could write the integrated margin on the right-hand side as

$M_{u1} + M_{d1}$ where $M_{u1} = w_1 - c_1$ and $M_{d1} = p_1 - w_1 - v_1$, and w_1 is the pre-merger wholesale price charged downstream firm 1. The foreclosure condition then becomes

$$\Delta\pi > 0 \qquad \text{if and only if} \qquad -\left(\frac{\Delta D_1}{\Delta D_2^d}\right) > \frac{M_{u2}}{M_{u1} + M_{d1}}.$$

The expression $\left(-\Delta D_1/\Delta D_2^d\right)$ can be broken into diversion and departure rate components. Let D_2 be the amount sold by the rival including sales that use inputs from other sources. We can then write

$$-\frac{\Delta D_1}{\Delta D_2^d} = \left(\frac{\Delta D_1}{-\Delta D_2}\right)\left(\frac{\Delta D_2}{\Delta D_2^d}\right) = Div_{21} \times Departure\ Rate$$

where $Div_{21} = \Delta D_1/-\Delta D_2$ is the diversion ratio from the rival to the integrated firm and $Departure\ Rate\ =\ \Delta D_2/\Delta D_2^d$ is the change in the unit sales of the foreclosed firm relative to the number of units foreclosed. This provides a rigorous definition of the departure rate for use in foreclosure arithmetic.

Substituting this expression into the expression for profitable foreclosure gives

$$\Delta\pi > 0 \qquad \text{if and only if} \qquad Div_{21} \times Departure\ Rate\ > \frac{M_{u2}}{M_{u1} + M_{d1}},$$

which is the foreclosure condition in the paper.

<u>Formal Derivation of vGUPPI</u>

The diversion component of the value of diverted sales in the vGUPPI also breaks naturally into two components: the diversion ratio, which is a property of the final demand system, and the departure rate, which depends on the intensity of upstream competition. Tethering the departure rate is just as critical in vGUPPI analysis as it is in vertical foreclosure math.

The pre-merger profit of the upstream firm is

$$\pi^{Pre} = (w_1 - c)D_1^d + (w_2 - c)D_2^d$$

where D_i^d is the residual derived demand for upstream input from downstream firm i $(i = 1,2)$. The residual derived demand for the input depends on w_1 and w_2 as well as competition from other upstream firms, which is left unspecified.

The first order condition for merged firm's choice of w_2 can be written in differential form as follows (where it is understood that the changes are driven by changes in w_2):

$$w_2^{Pre} = c + \frac{D_2^d}{-\Delta D_2^d} + (w_1 - c)\frac{\Delta D_1^d}{-\Delta D_2^d}.$$

After a merger between upstream firm and downstream firm 1, the merged firm has an additional term in its profit objective:

$$\pi^{Post} = (w_1 - c)D_1^d + (w_2 - c)D_2^d + (p_1 - w_1 - v_1)D_1$$

where D_1 is the final demand for product 1 and v_1 is the downstream marginal cost of selling D_1 due to inputs other than the upstream firm's input. The post-merger first order condition for the merged firm's profit maximizing choice of w_2, holding w_1 fixed and holding the merged firm's *incentives* in setting p_1 fixed, is the following:

$$w_2^{Post} = c + \frac{D_2^d}{-\Delta D_2^d} + (w_1 - c)\frac{\Delta D_1^d}{-\Delta D_2^d} + (p_1 - w_1 - v_1)\frac{\Delta \mathrm{D}_1}{-\Delta D_2^d}$$

In the third term, the factor ΔD_1 is the change in merged *downstream quantity* assuming that w_1 is fixed and the incentives that drive the setting of p_1 are unchanged in response to the merger, i.e., p_1 is the downstream equilibrium price in the absence of the merger given w_1 and w_2. This isolates the incentive effects of the merger on w_2 alone. It is true that this is a fiction. In reality, everything changes in moving from the pre-merger to the post-merger equilibrium, including the objective function that is relevant in setting w_1 and p_1. The analysis here isolates the incentive effect related to w_2, but one must recognize that other effects interact in equilibrium and that the net equilibrium effect would account for all effects.

Define the pricing pressure on w_2 due to the merger as the difference between w_2^{Post} and w_2^{Pre}:

$$PP_{w_2} = w_2^{Post} - w_2^{Pre} = (p_1 - w_1 - v_1)\frac{\Delta D_1}{-\Delta D_2^d}.$$

Let's unpack the pricing pressure expression. Pricing pressure for w_2 equals the value to the merged firm (the margin component) of the fraction of the upstream division's lost sales due to an increase in w_2 that switch to interated downstream firm. The difference between the switching fraction and an ordinary diversion ratio is that the denominator is the change in the upstream division's residual *derived demand* due to the increase in the wholesale price. Because the upstream division faces competition in the upstream market, some fraction of their lost sales may involve the downstream rival switching to other upstream suppliers. Those sales do not involve diversion to the integrated downstream division.

A useful rearrangement of PP_{w_2} is obtained by dividing both the numerator and denominator by $-\Delta D_2$, where D_2 is the sales of downstream rivals that use inputs supplied by the merging upstream firm and inputs supplied by rival upstream firms. This yields

$$PP_{w_2} = (p_1 - w_1 - v_1)\frac{\Delta D_1/(-\Delta D_2)}{-\Delta D_2^d/(-\Delta D_2)}$$

$$= (p_1 - w_1 - v_1)\ Div_{21} \times \frac{\Delta D_2}{\Delta D_2^d}$$

$$= (p_1 - w_1 - v_1) \times\ Div_{21} \times Departure\ Rate$$

where Div_{21} is the diversion ratio in the downstream market from downstream rivals to merged downstream firm, and the Departure rate is the fraction of sales lost by merged upstream division that result in a reduction of sales to downstream rivals, i.e., the fraction of lost upstream sales of the merging firm that have the potential to be diverted to merged downstream division. This expression shows that the direct incentive effect on the price charged to rivals is the integrated firm's downstream margin times the diversion ratio times the departure rate.

There are two other incentive effects to discuss. First, the merger lowers the merged downstream division's effective marginal cost from w_1 to c. Holding constant p_2 and the *incentives* for setting p_1, this puts downward pressure on p_1 equal to the reduction in firm 1's marginal cost, which is given by the pre-merger upstream margin on sales to firm 1. This is the EDM effect:[1]

$$PP_{EDM} = -(w_1 - c).$$

Second, there is a factor that works in the opposite direction of the EDM effect. Specifically, the merged firm's post-merger downstream sales cannibalize its input sales to downstream rivals, which is an opportunity cost from lowering p_1 in response to EDM. Denote the change in a variable due to a reduction in p_1 by Δ_1. The price pressure on the integrated firm's downstream price due to this opportunity cost is

$$PP_{Opp\ Cost} = (w_2 - c)\frac{\Delta_1 D_2^d}{-\Delta_1 D_1}$$

$$= (w_2 - c)\left(\frac{\Delta_1 D_2}{-\Delta_1 D_1}\right)\left(\frac{\Delta_1 D_2^d}{\Delta_1 D_2}\right)$$

$$= (w_2 - c) \times Div_{12} \times InpRec$$

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where Div_{12} is the diversion ratio from integrated downstream division to rivals and $InpRec$ is the share of customers diverted that become recaptured input sales for integrated firm.

If we consider EDM and the opportunity cost simultaneously (which this analysis effectively does by evaluating the opportunity cost effect at $w_1 = c$), the net effect on the integrated firm's downstream price (assuming the upstream firm gets its share of customers switching to rivals) is

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[1] The Vertical Merger Guidelines associate EDM with this effect.

$$PP_{p_1} = -(w_1 - c) + (w_2 - c) \times Div_{12} \times InpRec.$$

If the downstream firms pay the same wholesale price w pre-merger, the net EDM effect is

$$PP_{p1/Symmetry} = -(w - c)(1 - Div_{12} \times InpRec).$$

As noted earlier, an equilibrium model would consider all these effects simultaneously. A complete determination of the net effects of the merger would require such a model. It is known that a simple comparison or weighted average of the vGUPPI effect and the net EDM effect does not necessarily predict the net effect of the merger on consumers.

Tethering Departure – Downstream Bertrand Case

There are N^D downstream differentiated Bertrand competitors, a dominant upstream firm, and an upstream price-taking fringe. The dominant upstream firm sets wholesale prices to each downstream firm and takes the upstream fringe supply in response to those prices as given. To make the model consistent with assumptions that underly vGUPPI analysis, assume that downstream firms do not observe their rivals' wholesale prices and have passive beliefs when they observe an out-of-equilibrium offer, but they anticipate their rivals' wholesale prices correctly in equilibrium.

Some notation:

- w_i is the wholesale price paid by downstream firm i.
- v is the marginal cost of production downstream due to other inputs.
- c is the dominant upstream firm's marginal cost of production.
- p_i is the downstream price of firm i.
- p_{-i} is the vector of downstream prices of firms other than firm i.
- $S_i(w_i)$ is the fringe supply to downstream firm i.

Downstream firm i's profit is

$$\pi_i = (p_i - w_i - v)D_i(p_i, p_{-i}).$$

Downstream firm i's first order condition for profit maximization can be written

$$M_D \equiv p_i - w_i - v = \frac{D_i}{-\frac{\partial D_i}{\partial p_i}}.$$

Let $r_i(w_i, p_{-i})$ be the reaction function of firm i. The profit of the dominant upstream firm is

$$\pi_u = \sum_{i}^{N^D} (w_i - c)\left(D_i\big(r_1(w_1, p_{-1}), r_2(w_2, p_{-2}), \ldots, r_N(w_N, p_{-N})\big) - S_i(w_i)\right).$$

Assume demand is symmetric, i.e., $\partial D_i / \partial p_i = \partial D_j / \partial p_j$ for all $i \neq j$. Assume further that $\partial D_i / \partial p_j = -\gamma\, \partial D_i / \partial p_i$ for all $i \neq j$ where γ is the diversion ratio from i to j. Define $\rho = \partial r_i / \partial w_i$ as the pass-through rate. In a symmetric equilibrium, $w_i = w$ for all i, and the first order condition for the upstream firm's optimal choice of w_i can be written as

$$M_u \equiv w - c = \frac{D_i - S_i}{-\left[\left(\frac{\partial D_i}{\partial p_i}\right)(1 - (N^D - 1)\gamma)\rho - S_i{}'\right]}.$$

Dividing M_u by M_D and multiplying the numerator and denominator by $[1 - (N^D - 1)\gamma]$ gives

$$\frac{M_u}{M_D} = \left(\frac{D_i - S_i}{D_i}\right)\left[\frac{(\partial D_i / \partial p_i)[1 + (N^D - 1)\gamma]\rho}{(\partial D_i / \partial p_i)[1 + (N^D - 1)\gamma]\rho - S_i'}\right]\left(\frac{1}{(1 - (N^D - 1)\gamma)\rho}\right)$$

The term in square brackets is a measure of the *departure rate* associated with a small increase in w_i, that is, the change in downstream quantity demanded divided by the change in the dominant firm's input supply in response to a small increase in w_i. Solving for this "teathered" departure rate $DEP_{Tethered}$ gives

$$DEP_{Tethered} = \left(\frac{M_u}{M_D}\right)\left(\frac{1}{s_{DF}}\right)(1 - (N^D - 1)\gamma)\rho$$

where $s_{DF} = (D_i - S_i)/D_i$ is the dominant firm's share of input supplied to downstream firm i.

Observe that the tethered departure rate is proportional to the ratio of the upstream and downstream profit margins. Intuitively, the smaller this margin ratio, the more intense is upstream competition. Observe as well that the tethered departure rate is decreasing in the dominant firm's share of the input supplied downstream, which is just the opposite of what is often assumed in untethered foreclosure arithmetic, which sometimes posits that departure rate should rise and fall with the dominant upstream firm's share because a smaller share means more intense upstream competition. This intuition is wrong when departure is *tethered* to the margin ratio so that optimizing conditions rationalize pre-merger upstream and downstream margins. The higher the dominant firm's market share, the less competition it faces from the fringe, and the smaller the departure rate.

Although it is out-of-equilibrium analysis, we can consider foreclosure math modified to tether the departure rate to pre-merger margins. Under standard foreclosure math, foreclosure is profitable when

$$(DEP)(\gamma) > \frac{M_u}{M_u + M_D}.$$

Substituting the tethered departure rate, this becomes

$$\left(\frac{M_u}{M_D}\right)\left(\frac{1}{s_{DF}}\right)[1-(N^D-1)\gamma]\rho\gamma > \frac{M_u}{M_u+M_D}$$

$$\Leftrightarrow \rho\gamma > \left(\frac{\mathrm{M_D}}{\mathrm{M_u}+\mathrm{M_D}}\right)\frac{s_{DF}}{[1-(N^D-1)\gamma]}$$

$$\Leftrightarrow \rho\gamma > \left(1-\frac{M_U}{M_u+M_D}\right)\frac{s_{DF}}{1-(N^D-1)\gamma}$$

$$\Leftrightarrow \rho\gamma > (1-\text{Upstream Profit Share})\frac{s_{DF}}{1-(N^D-1)\gamma} \quad or$$

$$\text{Foreclosure profitable} \Leftrightarrow \text{Upstream Profit Share} > 1-\frac{\rho\gamma[1-(N^D-1)\gamma]}{s_{DF}}.$$

For illustration, assume: there are two downstream firms, $N^D = 2$; the pass-through rate is $\rho = .5$ (linear demand); the diversion ratio is $\gamma = .9$ (meaning 10% of diversion from either downstream firm goes to an outside good and the rest stays in the market); and the share of the dominant upstream firm is 50%, $s_{DF} = 0.5$. This illustration involves highly concentrated markets at both levels and plausible diversion. Under these assumptions, which are illustrative, foreclosure math with a tethered departure rate predicts that foreclosure is profitable when the upstream profit share exceeds 91% and is unprofitable otherwise.

Note that in general, the smaller the upstream profit share, the *less* likely it is that foreclosure will be profitable, which is just the opposite of predictions of standard foreclosure math.

<u>Tethered Depature -- Downstream Cournot Case</u>

The general point that a low upstream profit share implies that foreclosure is less likely is robust to the nature of downstream competition. This section establishes this point under downstream Cournot competition.

Let $P(Q)$ be inverse demand, $Q(p)$ be direct demand, and $Q = q_1 + \cdots + q_{N^D}$. The first order condition for profit maximization of downstream firm i is

$$p - w - v = P'q_i = \frac{q_i}{-Q'}.$$

Let $p^*(w)$ be the downstream equilibrium price given w. The first order condition of the dominant firm is

$$w - c = \frac{Q - S}{-[Q'\rho - S']}$$

where $\rho = p^{*\prime}(w)$ is a pass-through rate. Using the same shorthand as above for the margin ratio, we have

$$\frac{M_u}{M_D} = \left[\frac{Q'\rho}{Q'\rho - S'}\right]\left(\frac{Q - S}{Q}\right)\left(\frac{N^D}{\rho}\right).$$

The term in square brackets is a measure of departure in response to an increase in the wholesale price to all downstream firms prior to the merger. This is different than the response to an increase in the wholesale price charged only unintegrated downstream firms, but it is still informative for seeing how departure is tethered to relative margins. Solving for this tethered departure rate DEP^* (i.e., solving for the term in square brackets) gives

$$DEP^* = \left(\frac{M_u}{M_D}\right)\left(\frac{1}{s_{DF}}\right)\left(\frac{\rho}{N^D}\right)$$

Substituting DEP* into the expression for profitable foreclosure using standard foreclosure math gives

$$\text{Foreclosure is profitable} \iff \text{Upstream Profit Share} > 1 - \frac{\rho}{N^D s_{DF}}$$

Using the same illustration as in the Bertrand section ($\rho = .5$, $N^D = 2$, $s_{DF} = .5$), foreclosure is profitable when the upstream profit share exceeds 50% and is unprofitable otherwise.

The formulas in this and the preceding section illustrate show how foreclosure arithmetic can be applied in two standard cases.

The Numbers Equivalent Formula for the Margin Ratio

Under symmetric downstream competition, the first order condition for profit maximization can be written

$$P - v - w = -P'q_i = \frac{-P'Q}{N} = \frac{-Q}{NQ'} \tag{1}$$

where Q' indicates the derivative of the direct demand with respect to P (and where $Q(P)$ is the direct demand).

Let $P^*(w)$ be the equilibrium downstream price given the wholesale price w. Assuming that all quantities are measured in the same units, the demand for the upstream product is $Q\big(P^*(w)\big)$. The upstream firm's profit is then

$$\pi_U = (w - c)Q\big(P^*(w)\big).$$

The first order condition for the profit-maximizing wholesale price can be written

$$w - c = \frac{-Q}{Q'P^{*\prime}} = \frac{-Q}{Q'\rho} \qquad (2)$$

where $\rho = P^{*\prime}$ is the pass-through rate. Dividing (2) by (1) gives

$$\frac{w - c}{P - w - v} = \frac{N}{\rho}. \qquad (3)$$

This expression is the profit margin ratio expression in the paper. It says that if the upstream firm has monopoly power, the ratio of the upstream and downstream margins is the number of downstream competitors divided by the pass-through rate.

Although this expression was derived by assuming symmetric Cournot competition in the downstream market, the result applies more generally. The number of downstream competitors can be interpreted as the "numbers equivalent" of downstream firms, i.e., the number of equally-sized downstream Cournot competitors that would yield the observed margins under any assumption about downstream conduct. The interpretation is that whatever oligopoly game downstream firms are engaged in, their margins reflect the degree of competition we would see if there were N symmetric downstream Cournot competitors.